

# Product commuting maps with the $\lambda$ -Aluthge transform

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## Abstract

Let  $H$  and  $K$  be two Hilbert spaces and  $\mathcal{B}(H)$  be the algebra of all bounded linear operators from  $H$  into itself. The main purpose of this paper is to obtain a characterization of bijective maps  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  satisfying the following condition

$$\Delta_\lambda(\Phi(A)\Phi(B)) = \Phi(\Delta_\lambda(AB)) \quad \text{for all } A, B \in \mathcal{B}(H),$$

where  $\Delta_\lambda(T)$  stands the  $\lambda$ -Aluthge transform of the operator  $T \in \mathcal{B}(H)$ .

More precisely, we prove that a bijective map  $\Phi$  satisfies the above condition, if and only if  $\Phi(A) = UAU^*$  for all  $A \in \mathcal{B}(H)$ , for some unitary operator  $U : H \rightarrow K$ .

**Keywords:** Normal, Quasi-normal operators, Polar decomposition,  $\lambda$ -Aluthge transform.

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## 1. Introduction

Let  $H$  and  $K$  be two complex Hilbert spaces and  $\mathcal{B}(H, K)$  be the Banach space of all bounded linear operators from  $H$  into  $K$ . In the case  $K = H$ ,  $\mathcal{B}(H, H)$  is simply denoted by  $\mathcal{B}(H)$  which is a Banach algebra. For  $T \in \mathcal{B}(H, K)$ , we set  $\mathcal{R}(T)$  and  $\mathcal{N}(T)$  for the range and the null-space of  $T$ , respectively. We also denote by  $T^* \in \mathcal{B}(K, H)$  the adjoint operator of  $T$ .

The spectrum of an operator  $T \in \mathcal{B}(H)$  is denoted by  $\sigma(T)$  and  $W(T)$  is the numerical range of  $T$ .

An operator  $T \in \mathcal{B}(H, K)$  is a *partial isometry* when  $T^*T$  is an orthogonal projection (or, equivalently  $TT^*T = T$ ). In particular  $T$  is an *isometry* if  $T^*T = I$ , and *unitary* if  $T$  is a surjective isometry.

The polar decomposition of  $T \in \mathcal{B}(H)$  is given by  $T = V|T|$ , where  $|T| = \sqrt{T^*T}$  and  $V$  is an appropriate partial isometry such that  $\mathcal{N}(T) = \mathcal{N}(V)$  and  $\mathcal{N}(T^*) = \mathcal{N}(V^*)$ .

The Aluthge transform introduced in [1] as  $\Delta(T) = |T|^{\frac{1}{2}}V|T|^{\frac{1}{2}}$  to extend some properties of hyponormal operators. Later, in [11], Okubo introduced a more general notion called  $\lambda$ -Aluthge transform which has also been studied in detail.

For  $\lambda \in [0, 1]$ , the  $\lambda$ -Aluthge transform is defined by,

$$\Delta_\lambda(T) = |T|^\lambda V |T|^{1-\lambda}.$$

Notice that  $\Delta_0(T) = V|T| = T$ , and  $\Delta_1(T) = |T|V$  which is known as Duggal's transform. It has since been studied in many different contexts and considered by a number of authors (see for instance, [2, 3, 7, 8, 9, 12] and some of the references there). The interest of the Aluthge transform lies in the fact that it respects many properties of the original operator. For example,

$$\sigma_*(\Delta_\lambda(T)) = \sigma_*(T), \text{ for every } T \in \mathcal{B}(H), \quad (1)$$

where  $\sigma_*$  runs over a large family of spectra. See [7, Theorems 1.3, 1.5].

Another important property is that  $\text{Lat}(T)$ , the lattice of  $T$ -invariant subspaces of  $H$ , is nontrivial if and only if  $\text{Lat}(\Delta(T))$  is nontrivial (see [7, Theorem 1.15]).

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Recently in [4], F.Bothelho, L.Molnár and G.Nagy studied the linear bijective mapping on Von Neumann algebras which commutes with the  $\lambda$ -Aluthge transforms. They focus of bijective linear maps such that

$$\Delta_\lambda(\Phi(T)) = \Phi(\Delta_\lambda(T)) \text{ for every } T \in \mathcal{B}(H).$$

We are concerned in this paper with the more general problem of product commuting maps with the  $\lambda$ -Aluthge transform in the following sense,

$$\Delta_\lambda(\Phi(A)\Phi(B)) = \Phi(\Delta_\lambda(AB)) \text{ for every } A, B \in \mathcal{B}(H), \quad (2)$$

for some fixed  $\lambda \in ]0, 1[$ .

Our main result gives a complete description of the bijective map  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  which satisfies Condition (2) and is stated as follows.

**Theorem 1.1.** *Let  $H$  and  $K$  be a complex Hilbert spaces, with  $H$  of dimension greater than 2. Let  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  be bijective. Then,*

*$\Phi$  satisfies (2), if and only if, there exists an unitary operator  $U : H \rightarrow K$  such that*

$$\Phi(A) = UAU^* \quad \text{for all } A \in \mathcal{B}(H).$$

**Remark 1.1.** (1) *In one dimensional, the result of Theorem 1.1 fails, as given in the following example: let the map  $\Phi : \mathbb{C} \rightarrow \mathbb{C}$  defined by*

$$\Phi(z) = \begin{cases} \frac{1}{z} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

*Clearly  $\Phi$  is bijective and satisfies (2), but it is not additive.*

(2) *The map  $\Phi$  considered in our theorem is not assumed to satisfy any kind of continuity. However, an automatic continuity is obtained as a consequence.*

The proof of Theorem 1.1 is stated in next section. Several auxiliary results are needed for the proof and are established below.

## 2. Proof of the main theorem

We first recall some basic notions that are used in the sequel. An operator  $T \in \mathcal{B}(H)$  is normal if  $T^*T = TT^*$ , and is quasi-normal, if it commutes with  $T^*T$  ( i.e.  $TT^*T = T^*T^2$ ), or equivalently  $|T|$  and  $V$  commutes. In finite dimensional spaces every quasi-normal operator is normal. It is easy to see that if  $T$  is quasi-normal, then  $T^2$  is also quasi-normal, but the converse is false as shown by nonzero nilpotent operators .

Also, quasi-normal operators are exactly the fixed points of  $\Delta_\lambda$  (see [7, Proposition 1.10]).

$$T \text{ quasi-normal} \iff \Delta_\lambda(T) = T. \quad (3)$$

An idempotent self adjoint operator  $P \in \mathcal{B}(H)$  is said to be an orthogonal projection. Clearly quasi-normal idempotents are orthogonal projections.

Two projections  $P, Q \in \mathcal{B}(H)$  are said to be orthogonal if  $PQ = QP = 0$  and we denote  $P \perp Q$ . A partial ordering between orthogonal projections is defined as follows,

$$P \leq Q \text{ if } PQ = QP = P.$$

We start with the following lemma, which gives the "only if" part in our theorem. It has already been mentioned in other papers in the case  $H = K$  (see [3], for example). We give the proof for completeness.

**Lemma 2.1.** *Let  $U : H \rightarrow K$  be an unitary operator, and  $\lambda \in [0, 1]$ . We have the following identity*

$$\Delta_\lambda(UTU^*) = U\Delta_\lambda(T)U^*, \quad \text{for every } T \in \mathcal{B}(H).$$

*Proof.* Let  $T \in \mathcal{B}(H)$ . It is easy to check

$$|UTU^*| = U|T|U^* \quad \text{and} \quad |UTU^*|^\lambda = U|T|^\lambda U^*, \quad \lambda \in [0, 1].$$

Now, let  $T = V|T|$  be a polar decomposition. Then

$$UTU^* = UV|T|U^* = (UVU^*)(U|T|U^*) = \tilde{V}|UTU^*|,$$

where  $\tilde{V} = UVU^*$ .  $\tilde{V}$  is a partial isometry,  $\mathcal{N}(UTU^*) = \mathcal{N}(\tilde{V})$  and hence  $\tilde{V}|UTU^*|$  is the polar decomposition of  $UTU^*$ . This implies that :

$$\begin{aligned} \Delta_\lambda(UTU^*) &= |UTU^*|^\lambda \tilde{V} |UTU^*|^{1-\lambda} \\ &= U|T|^\lambda U^* \tilde{V} U|T|^{1-\lambda} U^* \\ &= U|T|^\lambda V|T|^{1-\lambda} U^* \\ &= U\Delta_\lambda(T)U^*. \end{aligned}$$

This completes the proof. ■

For  $x, y \in H$ , we denote by  $x \otimes y$  the at most rank one operator defined by

$$(x \otimes y)u = \langle u, y \rangle x \quad \text{for } u \in H.$$

It is easy to show that every rank one operator has the previous form and that  $x \otimes y$  is an orthogonal projection, if and only if  $x = y$  and  $\|x\| = 1$ . We have the following proposition,

**Proposition 2.1.** *Let  $x, y \in H$  be nonzero vectors. We have*

$$\Delta_\lambda(x \otimes y) = \frac{\langle x, y \rangle}{\|y\|^2} (y \otimes y) \quad \text{for every } \lambda \in ]0, 1[.$$

*Proof.* Denote  $T = x \otimes y$ , then  $T^*T = |T|^2 = \|x\|^2(y \otimes y) = \left(\frac{\|x\|}{\|y\|}(y \otimes y)\right)^2$  and  $|T| = \sqrt{T^*T} = \frac{\|x\|}{\|y\|}(y \otimes y)$ . It follows that  $|T|^2 = \|x\|\|y\||T|$  and  $|T|^\gamma = (\|x\|\|y\|)^{\gamma-1}|T|$  for every  $\gamma > 0$ .

Now, let  $T = U|T|$  be the polar decomposition of  $T$ , we have

$$\begin{aligned} \Delta_\lambda(T) &= |T|^\lambda U|T|^{1-\lambda} \\ &= (\|x\|\|y\|)^{\lambda-1} (\|x\|\|y\|)^{-\lambda} |T|U|T| \\ &= \frac{1}{\|x\|\|y\|} |T|T \\ &= \frac{1}{\|y\|^2} (y \otimes y) \circ (x \otimes y) = \frac{\langle x, y \rangle}{\|y\|^2} (y \otimes y). \end{aligned}$$

We deduce the next ■

**Corollary 2.1.** *Let  $R$  be a bounded linear operator on  $H$  and  $\lambda \in ]0, 1[$ . Suppose that*

$$\Delta_\lambda(RT) = \Delta_\lambda(TR),$$

*for every rank one operator of the form  $T = y \otimes y$ . Then, there exists some  $\alpha \in \mathbb{C}$  such that  $R = \alpha I$ .*

*Proof.* Denote  $A = R^*$ . First, we claim that the linear operator  $A$  satisfies the property that for every  $z \in H$  we either have  $Az$  is orthogonal to  $z$  (calling  $z$  being of the first kind) or  $Az, z$  are linearly dependent (calling  $z$  being of the second kind). Indeed, let  $z \in H$  and  $T = z \otimes z$ , from the assumption and the Proposition 2.1, we have

$$\langle Rz, z \rangle z \otimes z = \Delta_\lambda(Rz \otimes z) = \Delta_\lambda(z \otimes Az).$$

In the case when  $\langle Rz, z \rangle = 0$ , then  $z$  is of the first kind. And if  $\langle Rz, z \rangle \neq 0$  then  $Az \neq 0$ , and from the last equality it follows that

$$\langle Rz, z \rangle z \otimes z = \frac{\langle Rz, z \rangle}{\|Az\|^2} Az \otimes Az.$$

Thus  $Az$  and  $z$  are linearly dependent.

Now,  $A$  is a scalar multiple of the identity. Indeed, on contrary assume that we have vector  $x$  which is of the first kind but not of the second kind and that we have a vector  $y$  which is of the second kind but not of the first kind. Then  $x, y$  are linearly independent. We may assume that  $Ay = y$ . Set  $x' = Ax$ . For a real number  $t$  from the unit interval and for  $z_t = tx + (1-t)y$  we have  $Az_t = tx' + (1-t)y$ . It is clear that the equation  $\langle Az_t, z_t \rangle = t(1-t)(\langle x', y \rangle + \langle y, x \rangle) + (1-t)^2\|y\|^2 = 0$  has at most one solutions  $t_1 \in ]0, 1[$ . Also, with the fact that  $x, y$  are linearly independent, then  $Az_t, z_t$  are linearly independent for all  $t \in ]0, 1[$  except for at most one  $t \in ]0, 1[$ . So, for example, for small enough positive  $t$  the vector  $z_t$  does not of the first kind nor of the second.

This shows that either have that  $Az$  is orthogonal to  $z$  for all vectors  $z$  or we have  $Az, z$  are linearly dependent for all vectors  $z$ . In the first case we have that  $A = 0$ , in the second one  $A$  is a scalar multiple of the identity. In any way  $A$  is a scalar multiple of the identity. Thus  $R = A^* = \alpha I$  for some  $\alpha \in \mathbb{C}$ . ■

The following lemma, provides a criterion for an operator to be positive through its  $\lambda$ -Aluthge transform. It will play a crucial role in the proof of Theorem 1.1.

**Lemma 2.2.** *Let  $T \in \mathcal{B}(H)$  be an invertible operator. The following conditions are equivalent :*

- (i)  *$T$  is positive;*
- (ii) *for every  $\lambda \in [0, 1]$ ,  $\Delta_\lambda(T)$  is positive;*
- (iii) *there exists  $\lambda \in [0, 1]$  such that  $\Delta_\lambda(T)$  is positive.*

*In particular,  $\Delta_\lambda(T) = cI$  for some nonzero scalar  $c$ , if and only if  $T = cI$ .*

*Proof.* The implications (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) are trivial. It remains to show that (iii)  $\Rightarrow$  (i). Let us consider the polar decomposition  $T = U|T|$  of  $T$  and assume that  $\Delta_\lambda(T)$  is a positive operator. Since  $T$  invertible it follows that  $|T|^{1-\lambda}$  is invertible and  $U$  is unitary. We claim that  $U = I$ . Indeed, let us denote  $A = |T|^{2\lambda-1}$ , we have

$$\begin{aligned} AU &= |T|^{2\lambda-1}U \\ &= |T|^{\lambda-1}(|T|^\lambda U |T|^{1-\lambda})|T|^{\lambda-1} \\ &= |T|^{\lambda-1}\Delta_\lambda(T)|T|^{\lambda-1}. \end{aligned}$$

This follows that  $AU = |T|^{\lambda-1}\Delta_\lambda(T)|T|^{\lambda-1}$  is positive. In particular it is self adjoint. Thus  $AU = (AU)^* = U^*A$  and then  $UAU = A$ . Therefore  $(AU)^2 = A^2$ . It follows that  $AU = A$  since  $AU$  and  $A$  are positive. Thus  $U = I$  and this gives  $T = U|T| = |T|$  is positive. ■

**Remark 2.1.** *The assumption  $T$  is invertible is necessary in the previous lemma. Indeed, let  $T = x \otimes y$ , with  $x, y$  be nonzero independent vectors such that  $\langle x, y \rangle \geq 0$ . Using proposition 2.1, we get  $\Delta_\lambda(T)$  is positive while  $T$  is not.*

**Lemma 2.3.** *Let  $T \in \mathcal{B}(H)$  be an arbitrary operator and  $P \in \mathcal{B}(H)$  be an orthogonal projection. The following are equivalent :*

(i)  $\Delta_\lambda(TP) = T$  ;

(ii)  $TP = PT = T$  and  $T$  is quasi-normal.

*Proof.* The implication (ii)  $\Rightarrow$  (i) is obvious. We show the direct implication. Consider  $TP = U|TP|$  the polar decomposition of  $TP$ . Suppose that  $\Delta_\lambda(TP) = T$ , then

$$|TP|^\lambda U|TP|^{1-\lambda} = T \quad \text{and} \quad |TP|^{1-\lambda} U^*|TP|^\lambda = T^*. \quad (4)$$

It follows that

$$\mathcal{R}(T) \subseteq \mathcal{R}(|TP|^\lambda) \subseteq \overline{\mathcal{R}(|TP|^2)}$$

and

$$\mathcal{R}(T^*) \subseteq \mathcal{R}(|TP|^{1-\lambda}) \subseteq \overline{\mathcal{R}(|TP|^2)}.$$

In the other hand, we have  $|TP|^2 = PT^*TP = P|T|^2P$ . Thus  $\overline{\mathcal{R}(|TP|^2)} \subseteq \mathcal{R}(P)$ . Hence  $\mathcal{R}(T) \subset \mathcal{R}(P)$  and  $\mathcal{R}(T^*) \subset \mathcal{R}(P)$ . Which implies that  $PT = T$  and  $PT^* = T^*$ . Therefore

$$PT = TP = T \quad \text{and} \quad T \text{ is quasi-normal.}$$

■

**Proposition 2.2.** *Let  $\Phi$  be a bijective map satisfying (2). Then*

$$\Phi(0) = 0.$$

Moreover, there exists a bijective function  $h : \mathbb{C} \rightarrow \mathbb{C}$  such that:

(i)  $\Phi(\alpha I) = h(\alpha)I$  for all  $\alpha \in \mathbb{C}$ .

(ii)  $h(\alpha\beta) = h(\alpha)h(\beta)$  for all  $\alpha, \beta \in \mathbb{C}$ .

(iii)  $h(1) = 1$  and  $h(-\alpha) = -h(\alpha)$  for all  $\alpha \in \mathbb{C}$ .

*Proof.* For the first assertion, since  $\Phi$  is bijective, there exists  $A \in \mathcal{B}(H)$  such that  $\Phi(A) = 0$ . Therefore  $\Phi(0) = \Delta_\lambda(\Phi(A)\Phi(0)) = 0$ .

Let us show now that there exists a function  $h : \mathbb{C} \rightarrow \mathbb{C}$  such that  $\Phi(\alpha I) = h(\alpha)I$  for all  $\alpha \in \mathbb{C}$ . If  $\alpha = 0$  the function  $h$  is defined by  $h(0) = 0$  since  $\Phi(0) = 0$ . Now, suppose that  $\alpha$  is a nonzero scalar and denote by  $R = \Phi(\alpha I)$  in particular  $R \neq 0$ . From (2) it follows that

$$\Delta_\lambda(R\Phi(A)) = \Phi(\Delta_\lambda(\alpha A)) = \Delta_\lambda(\Phi(A\alpha I)) = \Delta_\lambda(\Phi(A)R), \quad (5)$$

for every  $A \in \mathcal{B}(H)$ . Since  $\Phi$  is onto, then  $\Delta_\lambda(RT) = \Delta_\lambda(TR)$  for every rank one operator of the form  $T = y \otimes y$  from  $\mathcal{B}(K)$ . Since  $R = \Phi(\alpha I)$  different from zero and by Corollary 2.1, there exists a nonzero scalar  $h(\alpha) \in \mathbb{C}$  such that  $R = \Phi(\alpha I) = h(\alpha)I$ . In the other hand,  $\Phi$  is bijective and its inverse  $\Phi^{-1}$  satisfies the same condition as  $\Phi$ . It follows that the map  $h : \mathbb{C} \rightarrow \mathbb{C}$  is well defined and it is bijective.

Moreover, using again Condition (2), we get

$$h(\alpha\beta)I = \Delta_\lambda(\Phi(\alpha\beta I)) = \Delta_\lambda(\Phi(\alpha I)\Phi(\beta I)) = h(\alpha)h(\beta)I,$$

for every  $\alpha, \beta \in \mathbb{C}$  and therefore  $h$  is multiplicative.

Since  $(h(1))^2 = h(1)$  and  $h$  is bijective with  $h(0) = 0$ , we obtain  $h(1) = 1$ . Similarly  $h(-1) = -1$ , thus  $h(-\alpha) = h(-1)h(\alpha) = -h(\alpha)$  for all  $\alpha \in \mathbb{C}$ . ■

As a direct consequence we have the following corollary :

**Corollary 2.2.** *Let  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  be a bijective map satisfying (2). Then*

(i)  $\Phi(I) = I$ .

- (ii)  $\Delta_\lambda \circ \Phi = \Phi \circ \Delta_\lambda$ . In particular,  $\Phi$  preserves the set of quasi-normal operators in both directions.
- (iii)  $\Phi(\alpha A) = h(\alpha)\Phi(A)$  for all  $\alpha$  and  $A$  quasi-normal.

The following lemma gives some properties of bijective maps satisfying (2).

**Lemma 2.4.** *Let  $\Phi$  be a bijective map satisfying (2). Then*

- (1)  $\Phi(A^2) = (\Phi(A))^2$  for all  $A$  quasi-normal.
- (2)  $\Phi$  preserves the set of orthogonal projections.
- (3)  $\Phi$  preserves the orthogonality between the projections ;

$$P \perp Q \Leftrightarrow \Phi(P) \perp \Phi(Q).$$

- (4)  $\Phi$  preserves the order relation on the set of orthogonal projections in the both directions ;

$$Q \leq P \Leftrightarrow \Phi(Q) \leq \Phi(P).$$

- (5)  $\Phi(P + Q) = \Phi(P) + \Phi(Q)$  for all orthogonal projections  $P, Q$  such that  $P \perp Q$ .
- (6)  $\Phi$  preserves the set of rank one orthogonal projections in the both directions.

*Proof.* (1) From (2), we have  $\Delta_\lambda((\Phi(A))^2) = \Phi(\Delta_\lambda(A^2))$  for every operator  $A$ . Let  $A$  be a quasi-normal operator; since  $\Phi$  preserves the set of quasi-normal operators, we get  $\Phi(A), \Phi(A^2), (\Phi(A))^2$  are quasi-normal. It follows from (3) that  $\Delta_\lambda(A^2) = A^2$  and  $\Delta_\lambda(\Phi(A^2)) = \Phi(A^2)$ . We deduce that

$$(\Phi(A))^2 = \Delta_\lambda((\Phi(A))^2) = \Phi(\Delta_\lambda(A^2)) = \Phi(A^2).$$

(2) Follows from the first assertion since orthogonal projections are quasi-normal.

(3) Assume that  $P, Q$  are orthogonal and denote  $N = \Phi(P)$  and  $M = \Phi(Q)$ . From (2) we have,  $\Delta_\lambda(MN) = \Delta_\lambda(NM) = 0$  and using [6, Theorem 4], we obtain

$$(MN)^2 = MNMN = 0 \quad \text{and} \quad (NM)^2 = NMNM = 0.$$

It follows that,

$$\|MN\|^2 = \|(MN)^*MN\| = \|NMN\| = \|(NMN)^2\|^{\frac{1}{2}} = \|NMNMN\|^{\frac{1}{2}} = 0$$

and similarly,  $NM = 0$ .

Finally  $\Phi$  preserves the orthogonality between the projections.

(4) Now, assume that  $Q \leq P$ , then  $PQ = QP = Q$ . By (2) we have

$$\Delta_\lambda(\Phi(Q)\Phi(P)) = \Phi(Q).$$

By Lemma 2.3, we get  $\Phi(Q)\Phi(P) = \Phi(P)\Phi(Q) = \Phi(Q)$  since  $\Phi(P)$  is an orthogonal projection. Therefore  $\Phi(Q) \leq \Phi(P)$ . Since  $\Phi$  is bijective and its inverse satisfies the same conditions as  $\Phi$ , hence  $\Phi$  preserves the order relation between the projections in both directions.

(5) Suppose that  $P, Q$  are orthogonal. We have  $P \leq P + Q$  and  $Q \leq P + Q$ . Which gives  $\Phi(P) \leq \Phi(P + Q)$  and  $\Phi(Q) \leq \Phi(P + Q)$ . From  $\Phi(P) \perp \Phi(Q)$ , it follows that

$$\Phi(P) + \Phi(Q) \leq \Phi(P + Q).$$

Since  $\Phi^{-1}$  satisfies the same assumptions as  $\Phi$ , we have

$$\begin{aligned} \Phi(P + Q) &= \Phi[\Phi^{-1}(\Phi(P)) + \Phi^{-1}(\Phi(Q))] \\ &\leq \Phi[\Phi^{-1}(\Phi(P) + \Phi(Q))] \\ &= \Phi(P) + \Phi(Q). \end{aligned}$$

Finally  $\Phi(P + Q) = \Phi(P) + \Phi(Q)$ .

(6) Let  $P = x \otimes x$  be a rank one projection. We claim that  $\Phi(P)$  is a non zero minimal projection. Indeed, let  $y \in K$  be an unit vector such that  $y \otimes y \leq \Phi(P)$ . Thus  $\Phi^{-1}(y \otimes y) \leq P$ . Since  $P$  is a minimal projection and  $\Phi^{-1}(y \otimes y)$  is a non zero projection, then  $\Phi^{-1}(y \otimes y) = P$ . Therefore  $\Phi(P) = y \otimes y$  is a rank one projection. ■

We now prove the following lemma which is needed in the proof of our result.

**Lemma 2.5.** *Let  $\Phi$  be a bijective map satisfying (2). Let  $P = x \otimes x, Q = x' \otimes x'$  be rank one projections such that  $P \perp Q$ . Then*

$$\Phi(\alpha P + \beta Q) = h(\alpha)\Phi(P) + h(\beta)\Phi(Q)$$

for every  $\alpha, \beta \in \mathbb{C}$ .

*Proof.* If  $\alpha = 0$  or  $\beta = 0$  the result is trivial. Suppose that  $\alpha \neq 0$  and  $\beta \neq 0$ . Clearly  $\alpha P + \beta Q$  is normal, hence  $\Phi(\alpha P + \beta Q)$  is quasi-normal. By Condition (2) we get

$$\begin{aligned} \Phi(\alpha P + \beta Q) &= \Delta_\lambda(\Phi(\alpha P + \beta Q)) \\ &= \Phi(\Delta_\lambda(\alpha P + \beta Q)) \\ &= \Phi\left(\Delta_\lambda((\alpha P + \beta Q)(P + Q))\right) \\ &= \Delta_\lambda(\Phi(\alpha P + \beta Q)\Phi(P + Q)) \\ &= \Phi(\alpha P + \beta Q)\Phi(P + Q). \end{aligned}$$

Since  $\Phi(P + Q) = \Phi(P) + \Phi(Q)$  is a an orthogonal projection, hence by Lemma 2.3

$$\begin{aligned} \Phi(\alpha P + \beta Q) &= \Phi(\alpha P + \beta Q)(\Phi(P) + \Phi(Q)) = (\Phi(P) + \Phi(Q))\Phi(\alpha P + \beta Q) \\ &= (\Phi(P) + \Phi(Q))\Phi(\alpha P + \beta Q)(\Phi(P) + \Phi(Q)). \end{aligned}$$

Denote by  $T = \Phi(\alpha P + \beta Q)$ . We write  $\Phi(x \otimes x) = y \otimes y$  and  $\Phi(x' \otimes x') = y' \otimes y'$  with  $y \perp y'$ , since  $\Phi$  preserves orthogonality and rank one projections. We have,

$$T = (y \otimes y + y' \otimes y')T(y \otimes y + y' \otimes y').$$

Hence

$$T = \langle Ty, y \rangle y \otimes y + \langle Ty', y' \rangle y' \otimes y' + \langle Ty, y' \rangle y' \otimes y + \langle Ty', y \rangle y \otimes y'. \quad (6)$$

We show that  $\langle Ty', y \rangle = \langle Ty, y' \rangle = 0$  by using (2)

$$\begin{aligned} \Delta_\lambda(\Phi(\alpha P + \beta Q)\Phi(P)) &= \Phi(\Delta_\lambda((\alpha P + \beta Q)P)) \\ &= \Phi(\alpha P) = h(\alpha)\Phi(P). \end{aligned}$$

In other terms, we write

$$\Delta_\lambda(Ty \otimes y) = \Delta_\lambda(y \otimes T^*y) = h(\alpha)y \otimes y.$$

Since  $h(\alpha) \neq 0$ , then  $T^*y \neq 0$ . By Proposition 2.1 follows that

$$\langle Ty, y \rangle y \otimes y = \frac{\langle y, T^*y \rangle}{\|T^*y\|^2} T^*y \otimes T^*y = h(\alpha)y \otimes y.$$

Therefore  $\langle Ty, y \rangle = h(\alpha)$  and  $T^*y = \overline{h(\alpha)}y$ . Using (6) we deduce

$$T^*y = \langle T^*y, y \rangle y + \langle T^*y, y' \rangle y' = \overline{h(\alpha)}y.$$

It follows that  $\langle Ty', y \rangle = \langle T^*y, y' \rangle = 0$ .

By similar arguments we get  $\langle Ty', y' \rangle = h(\beta)$  and  $\langle Ty, y' \rangle = 0$ . Again (6) implies that

$$\Phi(\alpha P + \beta Q) = T = h(\alpha)y \otimes y + h(\beta)y' \otimes y' = h(\alpha)\Phi(P) + h(\beta)\Phi(Q).$$

■

Now, we are in a position to prove our main result

**Proof of Theorem 1.1.** The "only if" part is an immediate consequence of Lemma 2.1.

We show the "if" part. Assume that  $\Phi : \mathcal{B}(H) \rightarrow \mathcal{B}(K)$  is bijective and satisfies (2). The proof of theorem is organized in several steps.

Step 1. For every  $A \in \mathcal{B}(H)$ , we have

$$\langle \Phi(A)y, y \rangle = h(\langle Ax, x \rangle) \text{ for all unit vectors } x, y \text{ such } \Phi(x \otimes x) = y \otimes y. \quad (7)$$

Let  $x, y \in H$  be unit vectors such that  $\Phi(x \otimes x) = y \otimes y$ . From (2), we obtain

$$\begin{aligned} \Delta_\lambda(\Phi(A)y \otimes y) &= \Delta_\lambda(\Phi(A)\Phi(x \otimes x)) \\ &= \Phi(\Delta_\lambda(A(x \otimes x))) \\ &= \Phi(\Delta_\lambda(Ax \otimes x)). \end{aligned}$$

Using Proposition 2.1, we get

$$\langle \Phi(A)y, y \rangle = \langle y \otimes y, \Phi(\langle Ax, x \rangle x \otimes x) \rangle = h(\langle Ax, x \rangle) y \otimes y.$$

It follows that

$$\langle \Phi(A)y, y \rangle = h(\langle Ax, x \rangle).$$

Step 2. The function  $h$  is additive.

Let  $P = x \otimes x, Q = x' \otimes x'$  are rank one projections such that  $P \perp Q$  and  $\alpha, \beta \in \mathbb{C}$ . Denote by  $z = \frac{1}{\sqrt{2}}(x + x')$ , then  $\|z\| = 1$  and  $\|Pz\|^2 = \|Qz\|^2 = \frac{1}{2}$ . Note  $z \otimes z$  is rank one projection, then there exist a unit vector  $u \in K$  such that  $\Phi(z \otimes z) = u \otimes u$ . We take  $A = \alpha P + \beta Q$  in the identity (7), we get that

$$\begin{aligned} \langle \Phi(\alpha P + \beta Q)u, u \rangle &= h(\langle \alpha Pz + \beta Qz, z \rangle) \\ &= h(\alpha \|Pz\|^2 + \beta \|Qz\|^2) \\ &= h(\frac{1}{2})h(\alpha + \beta). \end{aligned}$$

Thus

$$\langle \Phi(\alpha P + \beta Q)u, u \rangle = h(\frac{1}{2})h(\alpha + \beta). \quad (8)$$

In the other hand, by Lemma 2.5 we have

$$\Phi(\alpha P + \beta Q) = \Phi(\alpha P) + \Phi(\beta Q) = h(\alpha)\Phi(P) + h(\beta)\Phi(Q).$$

And therefore

$$\begin{aligned} \langle \Phi(\alpha P + \beta Q)u, u \rangle &= \langle \Phi(\alpha P)u + \Phi(\beta Q)u, u \rangle \\ &= \langle \Phi(\alpha P)u, u \rangle + \langle \Phi(\beta Q)u, u \rangle \\ &= h(\langle \alpha Pz, z \rangle) + h(\langle \beta Qz, z \rangle) \\ &= h(\alpha \|Pz\|^2) + h(\beta \|Qz\|^2) \\ &= h(\frac{1}{2})(h(\alpha) + h(\beta)). \end{aligned}$$

Using (8) and the preceding equality, it follows that

$$h(\frac{1}{2})h(\alpha + \beta) = h(\frac{1}{2})(h(\alpha) + h(\beta)).$$

Now  $h(\frac{1}{2}) \neq 0$  gives

$$h(\alpha + \beta) = h(\alpha) + h(\beta).$$



Step 3.  $h$  is continuous.

Let  $\mathcal{E}$  be a bounded subset in  $\mathbb{C}$  and  $A \in \mathcal{B}(H)$  such that  $\mathcal{E} \subset W(A)$ .

By (7),

$$h(\mathcal{E}) \subset h(W(A)) = W(\Phi(A))$$

Now,  $W(\Phi(A))$  is bounded and thus  $h$  is bounded on the bounded subset. With the fact that  $h$  is additive and multiplicative, it then follows that  $h$  is continuous (see, for example, [10]). We derive that  $h$  is a continuous automorphism over the complex field  $\mathbb{C}$ . It follows that  $h$  is the identity or the complex conjugation map.

Step 4. The map  $\Phi$  is linear or anti-linear.

Let  $y \in K$  and  $x \in H$  be two unit vectors, such that  $y \otimes y = \Phi(x \otimes x)$ . Let  $\alpha \in \mathbb{C}$  and  $A, B \in \mathcal{B}(H)$  be arbitrary. Using (7), we get

$$\begin{aligned} \langle \Phi(A+B)y, y \rangle &= h(\langle (A+B)x, x \rangle) \\ &= h(\langle Ax, x \rangle + \langle Bx, x \rangle) \\ &= h(\langle Ax, x \rangle) + h(\langle Bx, x \rangle) \\ &= \langle \Phi(A)y, y \rangle + \langle \Phi(B)y, y \rangle \\ &= \langle (\Phi(A) + \Phi(B))y, y \rangle, \end{aligned}$$

and

$$\langle \Phi(\alpha A)y, y \rangle = h(\langle \alpha Ax, x \rangle) = h(\alpha)h(\langle Ax, x \rangle) = h(\alpha) \langle \Phi(A)y, y \rangle.$$

Therefore

$$\langle \Phi(A+B)y, y \rangle = \langle (\Phi(A) + \Phi(B))y, y \rangle \quad \text{and} \quad \langle \Phi(\alpha A)y, y \rangle = h(\alpha) \langle \Phi(A)y, y \rangle,$$

for all unit vectors  $y \in K$ . It follows that  $\Phi(A+B) = \Phi(A) + \Phi(B)$  and  $\Phi(\alpha A) = h(\alpha)\Phi(A)$  for all  $A, B \in \mathcal{B}(H)$ . Therefore  $\Phi$  is linear or anti-linear since  $h$  is the identity or the complex conjugation.

Step 5. There exists an unitary operator  $U \in \mathcal{B}(H, K)$ , such that  $\Phi(A) = UAU^*$  for every  $A \in \mathcal{B}(H)$ .

Let  $A \in \mathcal{B}(H)$  be invertible. By (2), we have

$$\Delta_\lambda(\Phi(A)\Phi(A^{-1})) = \Delta_\lambda(\Phi(A^{-1})\Phi(A)) = \Phi(\Delta_\lambda(I)) = I.$$

By Lemma 2.2, we get that

$$\Phi(A)\Phi(A^{-1}) = \Phi(A^{-1})\Phi(A) = I.$$

It follows that  $\Phi(A)$  is also invertible and  $(\Phi(A))^{-1} = \Phi(A^{-1})$ . Therefore  $\Phi$  preserves the set of invertible operators. By [5, Corollary 4.3], there exists a bounded linear and bijective operator  $V : H \rightarrow K$  such that  $\Phi$  takes one of the following form

$$\Phi(A) = VAV^{-1} \quad \text{for all } A \in \mathcal{B}(H) \tag{9}$$

or

$$\Phi(A) = VA^*V^{-1} \quad \text{for all } A \in \mathcal{B}(H) \tag{10}$$

In order to complete the proof we have to show that  $V$  is unitary and  $\Phi$  has form (9).

First, we show that  $V : H \rightarrow K$  in (9) ( or in (10)) is necessarily unitary. Indeed, let  $x \in H$  be a unit vector. We know that  $x \otimes x$  is an orthogonal projection, hence  $\Phi(x \otimes x) = Vx \otimes (V^{-1})^*x$  is also an orthogonal projection. It follows that  $(V^{-1})^*x = Vx$  for all unit vector  $x \in H$  and then  $(V^{-1})^* = V$ . Therefore  $V$  is unitary.

Seeking contradiction, we suppose that (10) holds. Multiplying (10) by  $V^*$  left and by  $V$  right, since  $\Phi$  commutes with  $\Delta_\lambda$ , we obtain

$$\Delta_\lambda(A^*) = (\Delta_\lambda(A))^*, \quad \text{for every } A \in \mathcal{B}(H). \quad (11)$$

Let us consider  $A = x \otimes x'$  with  $x, x'$  are unit independent vectors in  $H$ .  $A^* = x' \otimes x$  and by Proposition 2.1, we have

$$\Delta_\lambda(A) = \langle x, x' \rangle (x' \otimes x') \quad \text{and} \quad \Delta_\lambda(A^*) = \langle x', x \rangle (x \otimes x),$$

which contradicts (11). This completes the proof.

### Acknowledgments.

I wish to thank Professor Mostafa Mbekhta for the interesting discussions as well as his useful suggestions for the improvement of this paper. Also, I thank the referee for valuable comments that helped to improve the paper, in particular the proof of Corollary 2.1.

This work was supported in part by the Labex CEMPI (ANR-11-LABX-0007-01).

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